PERGAMON

International Journal of Solids and Structures 39 (2002) 1031–1040

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Stochastic beam equations under random dynamic loads

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Received 13 September 2000

Abstract

The fourth order partial differential equation representing beams under random loading is considered. A general solution for this equation is obtained using the eigenfunction and variation of parameters techniques. Also the average and the variance of the beam deflection, shear and bending moment are obtained. The load is divided into a deterministic function and a randomly perturbed function representing the expected error in the deterministic load. The general closed form solution is obtained in stochastic integral terms. Some important statistical moments of the solution process are computed and illustrated. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords: Stochastic differential equation; Beam equation; Random loading; Eigenfunction expansion

1. Introduction

The differential equations (DEs) are influenced by random uncertainties in its differential operators, initial conditions and boundary conditions. Recently, such equations are the interest of many investigators, see for example Soong (1973) and Oksendal (1985). These equations are classified as stochastic ordinary DE or partial DE according to the nature of the models. The stochastic vibration equations are good examples of the first type, see Lin and Cai (1995) and Roberts and Spanos (1990) as examples. The stochastic diffusion equations and the stochastic wave equations are good examples of the second type, see El-Tawil (1996) and El-Tawil and Ebady (1999).

The beam equation is a fourth order partial DE and has very wide applications in structural engineering. As a DE, it has its own problems concerning existence, uniqueness and methods of solutions (Pipes and Harvill, 1970). As an engineering problem, it has its applications in beams, bridges and other structures (Clough and Penzien, 1975). Vanmarcke and Grigoriu (1983) discussed the analysis of shear beams of random rigidity. Spanos and Ghanem (1989) discussed the solution of problems involving material variability. The use of the stochastic finite element, SFEM, is famous in solving such problems (Baker et al., 1989; Behdinan et al., 1997; Haiato and Peng, 1998).

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PII: S0020-7683(01)00254-2

Analytically, Mahmoud and El-Tawil (1990, 1992) solved a beam resting on stochastic elastic supports. Recently, Elshakoff et al. (1999) presented solutions for four different classes of beam problems under deterministic or random loads.

In this paper, the general fourth order stochastic partial DE, known as the beam equation, is solved analytically in a general closed form expression depending on the governing parameters and under stochastically perturbed loads. The statistical moments of the beam deflection are computed in general closed form expressions. A case study is illustrated through some parametric studies.

2. General analysis

The general beam equation under random dynamic loading is known as the following (Paz, 1991):

$$\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 U(x,t)}{\partial x^2}\right)+\rho A\frac{\partial^2 U(x,t)}{\partial t^2}+C\frac{\partial U(x,t)}{\partial t}=F(x,t;\omega), \tag{1}$$

where A is the beam cross-section area, E the Young's modulus of elasticity, I the moment of inertia of beam cross-section, ρ the mass per unit volume of beam cross-section, C the damping coefficient, $U(x,t;\omega)$ the beam deflection, $F(x,t;\omega)$ the random dynamic load, in which ω: a random outcome of a triple probability space (Ω,κ,P), where Ω is a sample space, κ is σ-algebra associated with Ω, P is a probability measure. Fig. 1 shows the general shape of a beam.

Considering the case of a constant moment of inertia, dividing by ρA and introducing the following parameters:

$$c=\frac{C}{\rho A},\quad G(x,t;\omega)=\frac{F(x,t;\omega)}{\rho A},\quad \alpha^2=\frac{EI}{\rho A},$$

Eq. (1) takes the following form:

$$\alpha^2\frac{\partial^4 U(x,t)}{\partial x^4}+\frac{\partial^2 U(x,t)}{\partial t^2}+c\frac{\partial U(x,t)}{\partial t}=G(x,t;\omega) \tag{2}$$

For the simply supported beam the following boundary conditions are applied

$$U(0,t)=0, \tag{3}$$

$$\frac{\partial^2 U}{\partial x^2}(0,t)=0, \tag{4}$$

$$U(l,t)=0, \tag{5}$$

$$\frac{\partial^2 U}{\partial x^2}(l,t)=0, \tag{6}$$

where l is the beam length. The initial conditions are taken as general deterministic functions of x as the following:

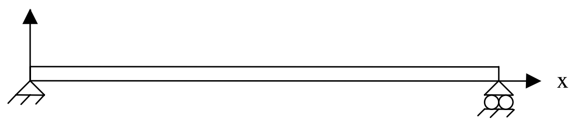


Fig. 1. The general shape of a simple beam.

$$U(x,0) = f(x), \tag{7}$$

$$\frac{\partial U}{\partial t}(x,0) = g(x). \tag{8}$$

Using the technique of eigenfunction expansion (Farlow, 1982) Eq. (2) has the following general solution:

$$U(x,t;\omega) = \sum_{n=1}^{\infty} T_n(t;\omega)\sin\frac{n\pi x}{l}. \tag{9}$$

Expanding the dynamic load $G(x,t;\omega)$ as sine Fourier series and substituting from Eq. (9) into Eq. (2), the following condition is obtained:

$$\ddot{T}_n(t) + c\dot{T}_n(t) + \alpha^2\left(\frac{n\pi}{l}\right)^4 T_n(t) = f_n(t;\omega), \tag{10}$$

where

$$f_n(t;\omega) = \frac{2}{l}\int_0^l G(x,t;\omega)\sin\frac{n\pi x}{l}\,\mathrm{d}x. \tag{11}$$

Using Eqs. (7) and (8), Eq. (10) is solved under the following initial conditions:

$$T_n(0) = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x, \tag{12}$$

$$\dot{T}_n(0) = \frac{2}{l}\int_0^l g(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x. \tag{13}$$

Using the technique of variation of parameters (Pipes and Harvill, 1970), Eq. (10) has the general solution:

$$T_n(t) = C_{1n}(t;\omega)T_{1n} + C_{2n}(t;\omega)T_{2n} \tag{14}$$

in which

$$\begin{aligned} C_{1n}(t;\omega) &= -\int \frac{T_{2n}(t)f_n(t;\omega)}{\eta_n(t)}\,\mathrm{d}t, \quad \eta_n(t) \neq 0, \\ C_{2n}(t;\omega) &= \int \frac{T_{1n}(t)f_n(t;\omega)}{\eta_n(t)}\,\mathrm{d}t, \end{aligned} \tag{15}$$

where

$$\eta_n(t) = T_{1n}(t)\dot{T}_{2n}(t) - \dot{T}_{1n}(t)T_{2n}(t). \tag{16}$$

The functions $T_{1n}(t)$ and $T_{2n}(t)$ are the homogeneous independent solutions of Eq. (10).

It has to be noted that the integrals in equalities (11), (14), and (15) are of a special type of integrals called a stochastic integral (McKean, 1969). Appendix A shows some important points on this subject in the mean square sense used in this paper.

3. Statistical moments of the solution process

The interest of this paper is to compute the ensemble average and the root m.s. error (the standard deviation) of the beam deflection analytically. This interest avoids the paper the problems coming out from the stochastic integrals theory. What really needed is the commutation property between the expectation operator E and the integration in m.s. sense, see Theorem A.2 in Appendix A.

Taking the ensemble average of the general solution given in Eq. (9), the average of the beam deflection takes the following form:

$$EU(x,t;\omega) = \sum_{n=1}^{\infty}[EC_{1n}(t;\omega)T_{1n}(t) + EC_{2n}(t;\omega)T_{2n}(t)]\sin\frac{n\pi x}{l} \tag{17}$$

in which

$$EC_{1n}(t;\omega) = -\int \frac{T_{2n}(t)Ef_n(t;\omega)\mathrm{d}t}{\eta_n(t)}, \tag{18}$$

where

$$Ef_n(t;\omega) = \frac{2}{l}\int_0^l EG(x,t;\omega)\sin\frac{n\pi x}{l}\mathrm{d}x. \tag{19}$$

The variance of the beam deflection is finally computed as the following:

$$\mathrm{Var}\,U(x,t;\omega) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\mathrm{Cov}(\xi_n,\xi_m), \tag{20}$$

where

$$\mathrm{Cov}(\xi_n,\xi_m) = E(\xi_n - E\xi_n)(\xi_m - E\xi_m) = \sum_{i=1}^{2}\sum_{j=1}^{2}T_{in}T_{jm}\mathrm{Cov}(C_{in},C_{jm}) \tag{21}$$

and

$$\xi_n = C_{1n}T_{1n} + C_{2n}T_{2n}. \tag{22}$$

The covariance term in Eq. (21) can be evaluated as the following:

$$\mathrm{Cov}(C_{1n},C_{1m}) = \int\int \frac{T_{2n}(h)T_{2m}(q)Ef_n(h)f_m(q)}{\eta_n(h)\eta_m(q)}\mathrm{d}h\,\mathrm{d}q - EC_{1n}EC_{1m} \tag{23}$$

and the other terms are evaluated in a similar manner. Now, the average function in the double integration of Eq. (23) can be evaluated as the following:

$$Ef_nf_m = \frac{4}{l^2}\int_0^l\int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi v}{l}EG(x,t;\omega)G(v,t;\omega)\,\mathrm{d}x\,\mathrm{d}v. \tag{24}$$

It is clear that the computations are related sequentially to the correlation function of the load.

It has to be noted that the analysis in the present paper assumes the existence of the load average and correlation. Also, the integrals appear in equalities are assumed to exist.

4. Case study

Practically, the stochastic uncertainty of the load can be constructed in a perturbation way. Let the stochastic part be as a perturbation function for the deterministic one, i.e.

$$G(x,t;\omega) = p(x,t) + \varepsilon\varphi(x,t;\omega), \tag{25}$$

where the stochastic process (s.p.) in equality (23) has a zero mean. This means that the average of the load is the deterministic part $p(x,t)$. The general solution approaches the deterministic one when the stochastic scale parameter ε tends to zero. The average of the beam deflection is computed through the same equation (17) and its consequences. Namely, Eq. (19) becomes:

$$Ef_n(t;\omega) = \frac{2}{l}\int_0^l p(x,t)\sin\frac{n\pi x}{l}\,dx. \tag{26}$$

The variance is computed using Eq. (20) and its consequences except for equality (24) which becomes:

$$Ef_nf_m = \frac{4}{l^2}\int_0^l\int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi v}{l}(p(x,t)p(v,t) + \varepsilon^2 E\varphi(x,t;\omega)\varphi(v,t;\omega))\,dx\,dv. \tag{27}$$

5. Illustrative examples

5.1. Illustrative example-I

To illustrate the results obtained in the previous case study, take a beam of short span bridge of length $l = 10$ m, $\rho A = 0.122$ t/m', $I = 92080 \times 10^{-8}$ m^4, Young's modulus $E = 2.1 \times 10^7$ t/m^2 then α^2 will be 158 498. Also let

$$p(x,t) = 4e^{-0.1t}\sin\frac{\pi x}{l} \tag{28}$$

and

$$\varphi(x,t;\omega) = 4e^{-0.1t}n(t;\omega)\sin\frac{\pi x}{l}, \tag{29}$$

where $n(t)$ is a time white noise which is known to have a zero mean and correlated as Dirac delta function (Arnold, 1974).

The initial displacement and velocity will be considered as:

$$f(x) = 0.01\sin\frac{\pi x}{l} \tag{30}$$

and

$$g(x) = 0.001 \text{ m/s}. \tag{31}$$

Performing the lengthy computations, the following results are obtained:

$$\begin{aligned} EU(x,t) = e^{-0.41t}\Big[&(0.02124e^{0.31t} + 0.01\cos 39.29t + 0.00013676\sin 39.29t)\sin\frac{\pi x}{l} \\ &+ 1.2001\times 10^{-2}\sin 353.64t\sin\frac{3\pi x}{l} + 2.592\times 10^{-3}\sin 982.32t\sin\frac{5\pi x}{l} \\ &+ 9.45\times 10^{-4}\sin 1925.35t\sin\frac{7\pi x}{l} + 4.44\times 10^{-4}\sin 3182.72t\sin\frac{9\pi x}{l} + \cdots\Big] \end{aligned} \tag{32}$$

and

$$\mathrm{Var}\,U(x,t;\omega) = 0.56169\varepsilon^2\sin^2\frac{\pi x}{l}e^{-0.2t}, \tag{33}$$

where only the first mode was used in the variance computations.The results are illustrated through Figs. 2–6.

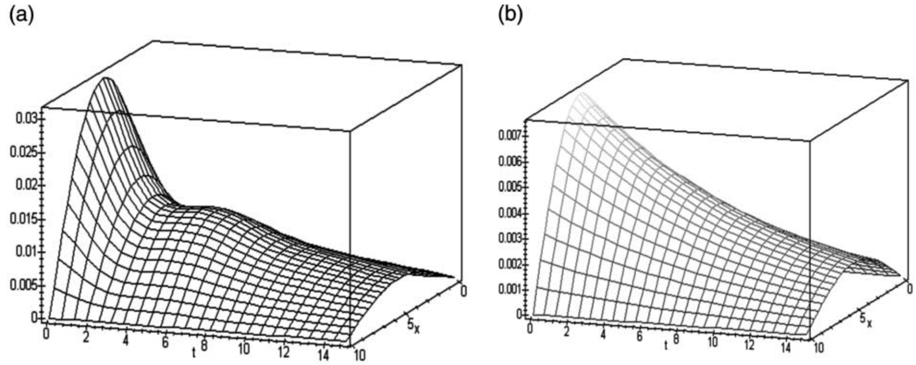


Fig. 2. Beam deflection with the span (x) and the time (t): (a) the first mode (due to deterministic part of the load) and (b) the RMS (due to stochastic part of the load).

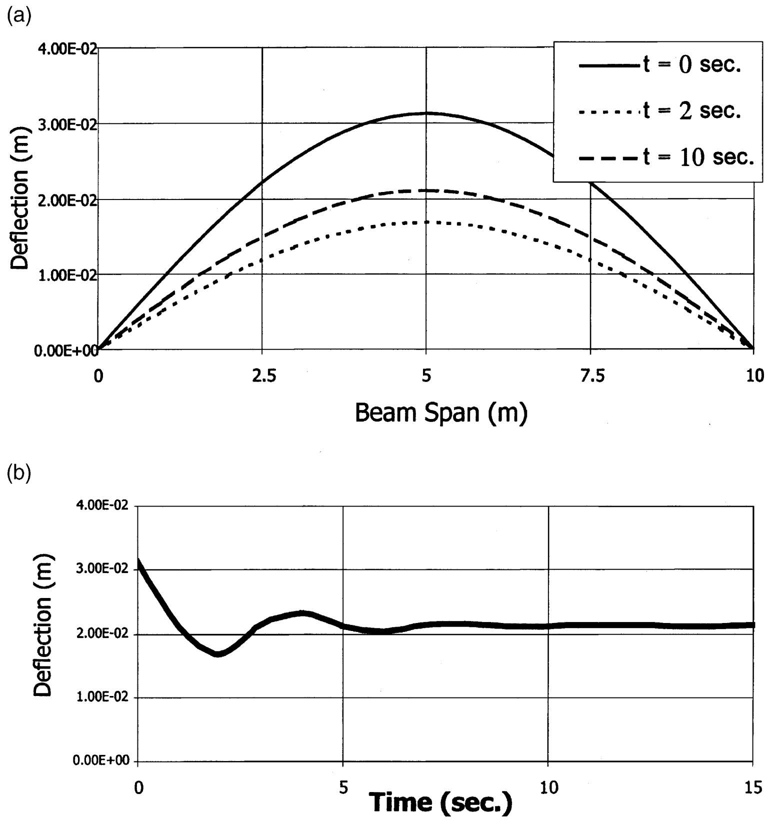
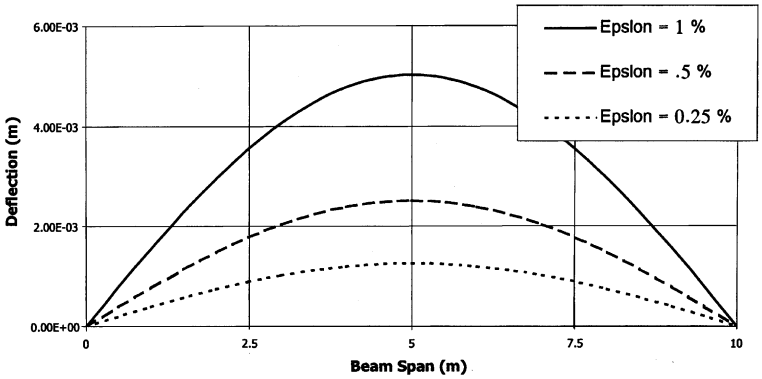


Fig. 3. The first mode of the beam deflection (due to deterministic part of the load) with (a) the span (x) in meters at different times and (b) the time (t) at mid-span.

(a)



(b)

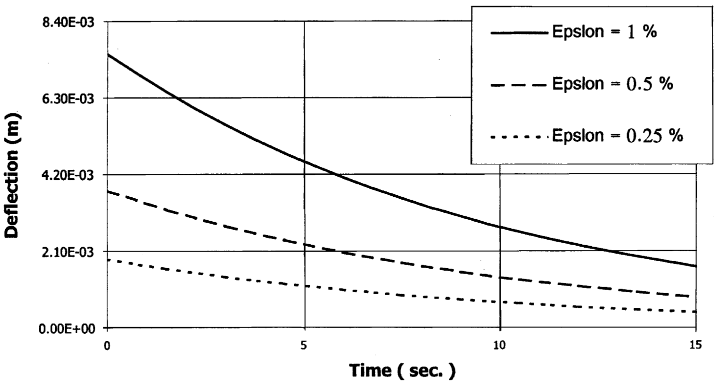


Fig. 4. The absolute value of beam deflection (due to stochastic part of the load) with (a) the span (x) for different values of Epslon and at time = 4 s and (b) the time (t) for different values of Epslon.

5.2. Illustrative example-II

In this example, more complicated load covariance function is considered to illustrate the analysis for a continuous load covariance function. Considering the same beam in the previous example and taking the load function as the sinusoidal process

$$\varphi(x,t;\omega) = \zeta \sin 2\pi t \text{ where } \zeta \text{ is uniformly distributed in } [0,1].$$

It can be shown that (Viniotis, 1998),

$$E\varphi(x,t;\omega) = \tfrac{1}{2}\sin 2\pi t, \qquad \mathrm{Var}\varphi(x,t;\omega) = \tfrac{1}{12}\sin^2 2\pi t$$

and

$$\mathrm{Cov}(t_1,t_2) = \tfrac{1}{12}\sin 2\pi t_1 \sin 2\pi t_2. \tag{34}$$

For the first mode,

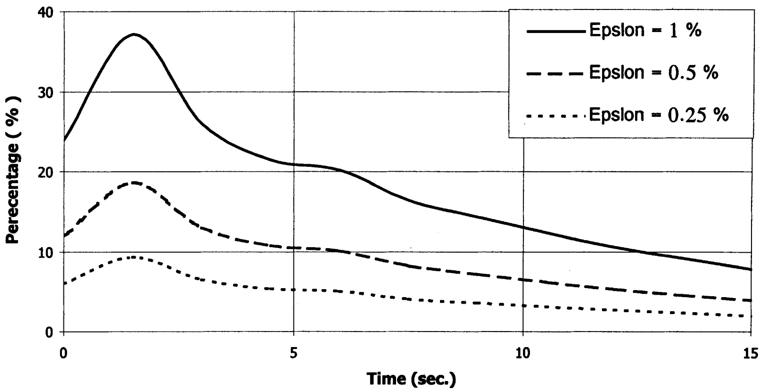


Fig. 5. Percentage of (stochastic deflection/deterministic deflection) with the time (t) for different values of Epslon and at mid-span.

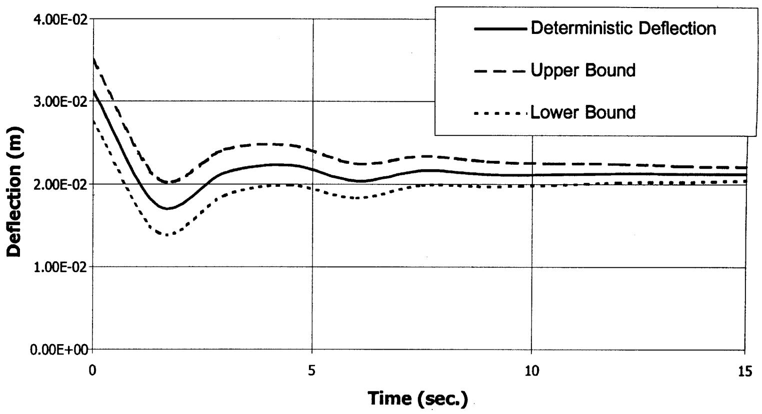


Fig. 6. Upper and lower bounds for mid-span deflection with time (t) for Epslon = 0.5%.

$$EC_{11}(t;\omega) = -0.1328122\,e^{-0.41t}(-0.01097076127\sin 45.572t - 0.00009870122268\cos 45.572t + 0.01514550619\sin 33.008t + 0.0001881258343\cos 33.008t)$$

$$EC_{21}(t;\omega) = -0.1328122\,e^{-0.41t}(-0.01097076127\cos 45.572t + 0.00009870122268\sin 45.572t + 0.01514550619\cos 33.008t - 0.0001881258343\sin 33.008t)$$

and

$$T_1(t) = e^{0.41t}\cos 39.29t, \qquad T_2(t) = e^{0.41t}\sin 39.29t.$$

Finally, the mean of the first mode beam deflection is

$$EU_1(x,t;\omega) = -0.1328122\sin\frac{\pi x}{l}(-0.000089424\cos 2\pi t + 0.0004174745\sin 2\pi t). \tag{35}$$

According to Eqs. (20) through (22), and after lengthy computations, the variance of the beam deflection is

$$\operatorname{Var} U(x,t;\omega) = \sin^2\frac{\pi x}{l}\{T_1(t)T_1(t)\operatorname{Cov}(C_{11},C_{11}) + T_1(t)T_2(t)\operatorname{Cov}(C_{11},C_{21}) + T_2(t)T_1(t)\operatorname{Cov}(C_{21},C_{11}) + T_2(t)T_2(t)\operatorname{Cov}(C_{21},C_{21})\},$$

where

$$\operatorname{Cov}(C_{11},C_{11}) = [-0.1328122\,\mathrm{e}^{-0.41t}(-0.01097076127\sin 45.572t - 0.00009870122268\cos 45.572t + 0.01514550619\sin 33.008t + 0.0001881258343\cos 33.008t)]^2,$$

$$\operatorname{Cov}(C_{21},C_{21}) = [-0.1328122\,\mathrm{e}^{-0.41t}(-0.01097076127\cos 45.572t + 0.00009870122268\sin 45.572t + 0.01514550619\cos 33.008t - 0.0001881258343\sin 33.008t)]^2,$$

$$\begin{aligned}\operatorname{Cov}(C_{11},C_{21}) &= \operatorname{Cov}(C_{21},C_{11})\\ &= [-0.1328122\,\mathrm{e}^{-0.41t}(-0.01097076127\sin 45.572t - 0.00009870122268\cos 45.572t\\ &\quad + 0.01514550619\sin 33.008t + 0.0001881258343\cos 33.008t)]\\ &\quad\times[-0.1328122\,\mathrm{e}^{-0.41t}(-0.01097076127\cos 45.572t + 0.00009870122268\sin 45.572t\\ &\quad + 0.01514550619\cos 33.008t - 0.0001881258343\sin 33.008t)].\end{aligned}$$

6. Results and conclusions

From the illustrative example-I, Fig. 3 shows that the root m.s. error is maximum at the mid span and cannot be ignored in the design of the beam when considering the random uncertainty of the load.

The vibrations of the mean function of the beam deflection died with time and attain its maximum at the mid span as expected as shown in Fig. 2. This is due to the mathematical nature of the load equation (28).

Appendix A. On stochastic integrals

A second order s.p. $X(t;\omega)$ is characterized by that its m.s. is finite, i.e.

$$\|X(t)\|^2 = EX^2(t;\omega) < \infty, \quad t \in T \tag{A.1}$$

and the sequence $\{X_n(t;\omega), t\in T\}$ of second order s.p. converges to a second order s.p. on T iff the correlation functions $E\{X_n(t;\omega)X_m(s;\omega), t\in T\}$ converge to a finite function $EX(t;\omega)X(s;\omega)$ as $n,m\to\infty$ in any manner whatever (Soong, 1973, p. 89). Also, a second order s.p. is m.s. continuous iff its correlation function is continuous.

Theorem A.1.

The s.p. $Y(u) = \int_a^b f(t,u)X(t;\omega)\,\mathrm{d}t$ $\exists$ iff the ordinary double Riemann integral

$$\int_a^b\int_a^b f(t,u)f(s,u)\Gamma_{XX}(t,s)\,\mathrm{d}t\,\mathrm{d}s$$

exists and is finite, where $\Gamma_{XX}(t,s)$ is the correlation function of $X(t)$ (Soong, 1973, p. 100).

Theorem A.2.

$$\text{If } Y(u) = \int_a^b f(t,u)X(t;\omega)\,\mathrm{d}t \quad \exists \text{ then}$$

$$EY(u) = \int_a^b f(t,u).EX(t;\omega)\,\mathrm{d}t,$$

also,

$$\Gamma_{YY}(u,v) = \int_a^b \int_a^b f(t,u)f(s,v)\Gamma_{XX}(t,s)\,\mathrm{d}t\,\mathrm{d}s,$$

Soong (1973, p. 104).

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